Compito di Metodi Matematici per l'Ingegneria

11/06/2018

1) $f(z) = \dfrac{\sin(\pi z)}{z^3(z-1)}$

$f(z): \mathbb{C} \setminus \{0,1\} \to \mathbb{C}$

$z=0$, $z=1$ annullano numeratore e denominatore. Le relazioni tra zeri e poli non possono essere applicate.

$$\lim_{z\to 1} \frac{\sin(\pi z)}{z^3(z-1)} = \frac{0}{0} \overset{(H)}{=} \lim_{z\to 1} \frac{\pi\cos(\pi z)}{3z^2(z-1)+z^3} = -\pi$$

$\Rightarrow$ $z=1$ singolarità eliminabile

$$\lim_{z\to 0} \frac{\sin(\pi z)}{z^3(z-1)} = \frac{0}{0} = \lim_{z\to 0} \underbrace{\frac{\sin(\pi z)}{\pi z}}_{\to 1} \frac{\pi}{z^2(z-1)} = \infty$$

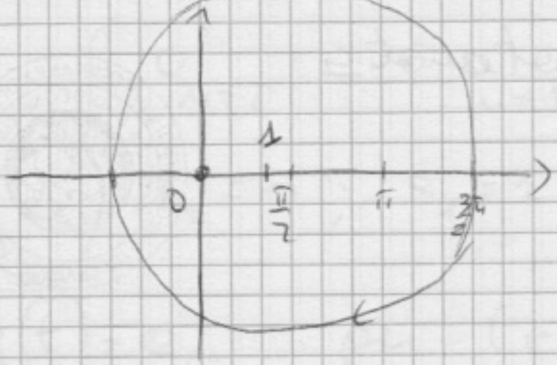
$\Rightarrow$ $z=0$ polo

$$\lim_{z\to 0} \frac{\sin(\pi z)}{z^3(z-1)}\, z = \lim_{z\to 0} \frac{\sin(\pi z)}{\pi z} \frac{\pi}{z(z-1)} = \infty$$

$$\lim_{z\to 0} \frac{\sin(\pi z)}{z^3(z-1)}\, z^2 = \lim_{z\to 0} \frac{\sin(\pi z)}{\pi z} \frac{\pi}{z-1} = -\pi$$

$\Rightarrow$ $z=0$ polo di ordine 2

$$\int_{+\gamma\left(\frac{\pi}{2},0\right)} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \underbrace{\operatorname*{Res}_{z=1} f(z)}_{=0} \right],$$ poiché



$z=0$ e $z=1$ interne al dominio T, che ha per frontiera $\gamma_{\pi}\left(\frac{\pi}{2}, \ldots\right)$

$\operatorname{Res}_{z=1} f(z) = 0$ perché $z=1$ è singolarità eliminabile

$$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} D\left[\frac{\sin(\pi z)}{z^3(z-1)}\, z^2\right] = \lim_{z\to 0} D\left[\frac{\sin(\pi z)}{z(z-1)}\right]$$

$$= \lim_{z\to 0} \frac{\pi\cos(\pi z) z(z-1) - \left[(z-1)+z\right]\sin(\pi z)}{z^2(z-1)^2} = \frac{0}{0} \overset{(H)}{=}$$

$$= \lim_{z\to 0} \frac{-\pi^2 \sin(\pi z)(z^2-z) + \cancel{\pi\cos(\pi z)\left[z-1+z\right]} - \cancel{\pi\cos(\pi z)\left[z-1+z\right]} - 2\sin(\pi z)}{2z(z-1)^2 + 2z^2(z-1)}$$

$$= \lim_{z\to 0} \frac{-\pi^2 \sin(\pi z)(z^2-z) - 2\sin(\pi z)}{2z(z-1)^2 + 2z^2(z-1)} = \frac{0}{0} \overset{(H)}{=}$$

$$= \lim_{z\to 0} \frac{-\pi^3 \cos(\pi z)(z^2-z) - \pi^2 \sin(\pi z)(2z-1) - 2\pi\cos(\pi z)}{2(z-1)^2 + 4z(z-1) + 4z(z-1) + 2z^2} =$$

$$= \frac{-2\pi}{2} = -\pi$$

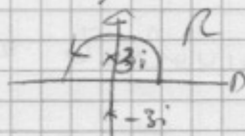
Quindi

$$\int_{+\gamma_{\pi}\left(\frac{\pi}{2}, \ldots\right)} f(z)\, dz = 2\pi i(-\pi) = -2\pi^2 i$$

2) $$\int_{-\infty}^{+\infty} \frac{x \sin(2x)}{x^2+9}\,dx$$ è l'int. improprio esiste

Consideriamo $f(z) = \dfrac{z\, e^{2iz}}{z^2+9} : \mathbb{C}\setminus\{\pm 3i\} \to \mathbb{C}$

olomorfa



$$\int_{-\infty}^{+\infty} \frac{x\, e^{2ix}}{x^2+9}\,dx = 2\pi i \operatorname*{Res}_{z=3i} f(z) - \lim_{R\to+\infty} \int_{+\gamma_R} \frac{z\, e^{2iz}}{z^2+9}\,dz$$

$$\operatorname*{Res}_{z=3i} f(z) = \frac{(3i)\, e^{-6}}{(2z)_{z=3i}} = \frac{e^{-6}}{2}$$

per la relazione tra zeri e poli

Invece per il lemma di Jordan, poiché

$$\varphi(z) = \frac{z}{z^2+9} \xrightarrow[z\to\infty]{} 0 \Rightarrow \lim_{R\to+\infty} \int_{+\gamma_R} \varphi(z)\, e^{2iz}\,dz = 0.$$

Quindi

$$\int_{-\infty}^{+\infty} \frac{x\, e^{2ix}}{x^2+9}\,dx = 2\pi i\, \frac{e^{-6}}{2} = \pi i\, e^{-6}$$

$$\int_{-\infty}^{+\infty} \frac{x \sin(2x)}{x^2+9}\,dx = \operatorname{Im}\left\{ \int_{-\infty}^{+\infty} \frac{x\, e^{2ix}}{x^2+9}\,dx \right\} = \frac{\pi}{e^6}$$

3) $\begin{cases} a_{n+2} - 6a_{n+1} + 9a_n = 5 \cdot 4^n \\ a_0 = 0 \ ; \ a_1 = 1 \end{cases}$

Poniamo

$y(t) = a_n$ per $t \in [n, n+1[$. Il problema è equivalente a

$$\begin{cases} y(t+2) - 6y(t+1) + 9y(t) = 5 \cdot 4^{[t]} \\ y(t) = 0 \text{ in } [0,1[\ ; \ y(t) = 1 \text{ in } [1,2[\end{cases}$$

Applichiamo ~~$Z_1[y(t)]$~~ la trasformata Z_1 e poniamo $Z_1[y(t)](z) = \mathcal{Y}$

$$Z_1[y(t+2)](z) - 6Z_1[y(t+1)](z) + 9Z_1[y(t)](t) =$$

$$= 5Z_1[4^{[t]}](z)$$

Si ha

$$Z_1[y(t+2)](z) = z^2\left[\mathcal{Y} - a_0 - \frac{a_1}{z}\right] = z^2\mathcal{Y} - z$$

$$Z_1[y(t+1)](z) = z[\mathcal{Y} - a_0] = z\mathcal{Y}$$

$$Z_1[4^{[t]}](z) = \sum_{n=0}^{+\infty} \left(\frac{4}{z}\right)^n = \frac{1}{1-\frac{4}{z}} = \frac{z}{z-4} \quad |z|>4$$

Quindi

$$z^2\mathcal{Y} - z - 6z\mathcal{Y} + 9\mathcal{Y} = 5\frac{z}{z-4}$$

$$(z^2 - 6z + 9)\mathcal{Y} = \frac{5z}{z-4} + z$$

$$\mathcal{Y} = \frac{5z + z^2 - 4z}{(z-4)(z-3)^2}$$

$$Z = \frac{z^2 + z}{(z-4)(z-3)^2}$$

Antitrasformando

$$a_n = \underset{z=4}{\text{Res}} \frac{z^{n-1}(z^2+z)}{(z-4)(z-3)^2} + \underset{z=3}{\text{Res}} \frac{z^{n-1}(z^2+z)}{(z-4)(z-3)^2}$$

$$\underset{z=4}{\text{Res}} \frac{z^{n}(z+1)}{(z-4)(z-3)^2} \overset{\text{per la relazione tra i poli}}{=} \frac{5\cdot 4^n}{\left[(z-3)^2 + 2(z-4)(z-3)\right]_{z=4}} =$$

$$= 5\cdot 4^n$$

per la

$$\underset{z\to 3}{\text{Res}} \frac{z^{n}(z+1)}{(z-4)(z-3)^2} = \lim_{z\to 3} D\left[\frac{z^n(z+1)\cancel{(z-3)^2}}{(z-4)\cancel{(z-3)^2}}\right]$$

$z=3$ è polo di ordine 2

$$= \lim_{z\to 3} \frac{\left(n z^{n-1}(z+1) + z^n\right)(z-4) - z^n(z+1)}{(z-4)^2} =$$

$$= \left(4n\, 3^{n-1} + 3^n\right)(-1) - 4\cdot 3^n = 3^n(-1-4) - 4n\, 3^{n-1}$$

$$= -5\cdot 3^n - 4n\, 3^{n-1}$$

Concludendo $\quad e_n = 5\cdot 4^n - 5\cdot 3^n - 4n\, 3^{n-1}$

2) a) $M/M/2$

$\lambda = 30/h \qquad \frac{1}{\mu} = 3' \qquad \mu = \frac{1}{3}\ 1/\text{min} = 20/h$

$$\rho = \frac{\lambda}{\mu\cdot 2} = \frac{30}{40} = \frac{3}{4} < 1$$

b) $$\pi_0 = \frac{1}{\sum_{n=0}^{1} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^2}{2(1-\rho)}} =$$

$$= \frac{1}{1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 \frac{1}{2 \cdot 1/4}} = \frac{1}{1 + \frac{3}{2} + \frac{9}{\cancel{4}} \frac{\cancel{4}}{2}} =$$

$$= \frac{1}{7}$$

c) $$L_q = \frac{\left(\frac{\lambda}{\mu}\right)^2 \pi_0 \, \rho}{2(1-\rho)^2} = \frac{\left(\frac{3}{2}\right)^2 \frac{1}{7} \frac{3}{4}}{2\left(\frac{1}{4}\right)^2} =$$

$$= \frac{\frac{9}{4} \frac{1}{7} \frac{3}{4}}{2 \frac{1}{\cancel{16}}} = \frac{27}{14}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\frac{27}{14}}{30} = \frac{9}{140} h = 0{,}06 h =$$

$$= 3{,}85'$$