08/01/2018

$$f(z) = \frac{z^2-4}{z \sin(\pi z)}$$

$\sin(\pi z) = 0 \iff \pi z = k\pi \quad k \in \mathbb{Z}$

$f(z): \mathbb{C} \setminus \{k,\ k \in \mathbb{Z}\} \to \mathbb{C} \qquad \iff z = k,\ k \in \mathbb{Z}$

Classifichiamo $z=0$

$\lim_{z\to 0} f(z) = \infty \Rightarrow z=0$ polo

Ordine del polo

$$\lim_{z\to 0} f(z)\cdot z = \lim_{z\to 0} \frac{z^2-4}{\sin(\pi z)} = \infty$$

$$\lim_{z\to 0} f(z)\cdot z^2 = \lim_{z\to 0} \frac{z(z^2-4)}{\sin(\pi z)} =$$

$$= \lim_{z\to 0} \frac{\pi z\,(z^2-4)}{\pi \sin(\pi z)} = -\frac{4}{\pi} \Rightarrow z=0 \text{ polo di ordine } 2$$

Classifichiamo $z=k,\ k\neq 0,\ k\in\mathbb{Z},\ k\neq \pm 2$

$$\lim_{\substack{z\to k\\ k\neq 0\\ k\neq\pm 2}} f(z) = \infty \Rightarrow z=k,\ (k\neq 0,\ k\neq\pm 2) \text{ polo}$$

$$\lim_{\substack{z\to k\\ k\neq 0\\ k\neq\pm 2}} \frac{z^2-4}{z\sin(\pi z)}(z-k) = \frac{k^2-4}{k} \lim_{\substack{z\to k\\ k\neq 0,\pm 2}} \frac{z-k}{\sin(\pi z)} = \frac{0}{0} =$$

$$= \frac{k^2-4}{k} \lim_{\substack{z\to k\\ k\neq 0,\pm 2}} \frac{1}{\pi\cos(\pi z)} = \frac{k^2-4}{k\pi}\,\frac{1}{\cos(k\pi)} = \Rightarrow$$

$$= \operatorname*{Res}_{z=k,\ k\neq 0,\pm 2} f(z)$$

$z=k, \quad k\neq 0, \pm 2, \quad$ poli di ordine 1

Classifichiamo $z=\pm 2$

$$\lim_{z\to 2} f(z) = \frac{0}{0} = \lim_{z\to 2} \frac{2z}{\sin(\pi z)+\pi z\cos(\pi z)} = \frac{4}{2\pi} =$$

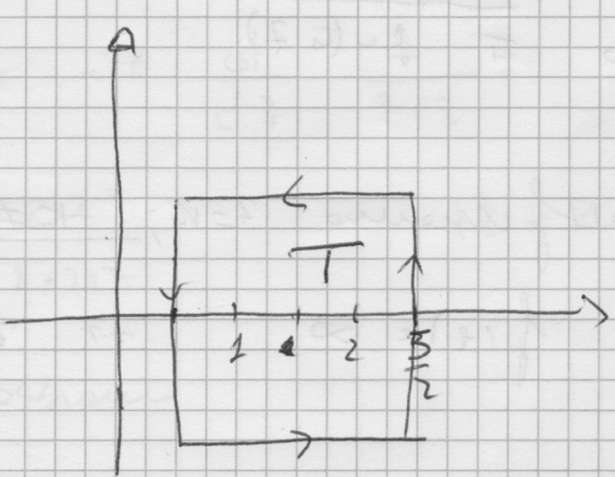
$$= \frac{2}{\pi} \quad \Rightarrow \quad z=2 \text{ singolarità apparente}$$

$$\lim_{z\to -2} f(z) = \frac{0}{0} = \lim_{z\to -2} \frac{2z}{\sin(\pi z)+\pi z\cos(\pi z)} = \frac{-4}{-2\pi} = \frac{2}{\pi}$$

$\Rightarrow \quad z=-2$ singolarità apparente

Quindi $\begin{cases} z=0 & \text{polo di ordine 2} \\ z=\pm 2 & \text{singolarità apparente} \\ z=k, \quad k\neq 0,\pm 2 & \text{poli di ordine 1} \end{cases}$

$$\int_{+\gamma} f(z)\,dz =$$



Le singolarità $z=k, \ k\in\mathbb{Z}$ sono tutte esterne a T, tranne $z=1$ e $z=2$. Per il Teorema dei residui

$$\int_{+\gamma} f(z)\,dz = 2\pi i\left[\operatorname*{Res}_{z=1} f(z) + \underbrace{\operatorname*{Res}_{z=2} f(z)}_{=0}\right] = 2\pi i\left[\frac{1-4}{\pi}\cdot\frac{1}{\cos\pi}\right]$$

$$= 6i$$

2) $\displaystyle\int \frac{x e^{ix}}{(x^2+1)^2}\, dx$

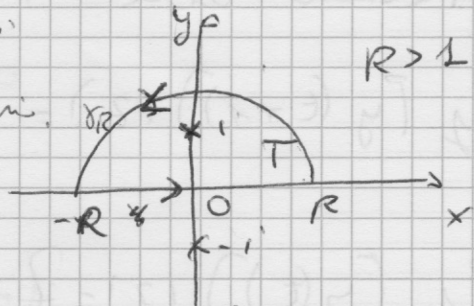
$\text{grad } B(x) \geq \text{grad } A(x) + 1 \Rightarrow \exists$ l'integrale improprio

Consideriamo $\quad f(z) = \dfrac{z e^{iz}}{(z^2+1)^2} : \mathbb{C} \setminus \{\pm i\} \to \mathbb{C}$

Per la relazione tra zeri e poli, $z = \pm i$ poli di ordine 2

Applichiamo il Teorema dei residui alla curva $\gamma = \gamma_R \cup [-R,R]$ percorsa in verso antiorario.

$z = i$ è interno a T, $z = -i$ esterno a T.



$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \operatorname*{Res}_{z=i} \frac{z e^{iz}}{(z^2+1)^2} - \lim_{R \to +\infty} \int_{+\gamma_R} f(z)\,dz$$

Per il lemma di Jordan, poiché $\varphi(z) = \dfrac{z}{(z^2+1)^2} \to 0$ al tendere di $z \to \infty \Rightarrow \displaystyle\lim_{R \to +\infty} \int_{+\gamma_R} \frac{z e^{iz}}{(z^2+1)^2}\,dz = 0$

Quindi

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \lim_{z \to i} D\left[\frac{z e^{iz}}{(z^2+1)^2}(z-i)^2\right] =$$

$$= 2\pi i \lim_{z \to i} D \frac{z e^{iz}}{(z+i)^2} = 2\pi i \lim_{z \to i} \frac{\left[e^{iz} + i z e^{iz}\right](z+i)^2 - 2(z+i) z e^{iz}}{(z+i)^4} =$$

$$= 2\pi \frac{1}{4e} = \frac{\pi i}{2e}$$

3) $\begin{cases} a_{n+1} + 2a_n = n(-2)^n \\ a_0 = 0 \end{cases}$

Passiamo all'equazione alle differenze, ponendo $y(t) = a_n$ per $t \in [n, n+1[$:

$$\begin{cases} y(t+1) + 2y(t) = [t](-2)^{[t]} \quad (*) \\ y(t) = 0 \quad \text{in } [0,1[\end{cases}$$

Passiamo alla trasformata Z di parametro 1:

$$Z_1[y(t+1)](z) = z\left[Z_1(y(t))(z) - a_0 \right] = z\mathcal{Z}$$

$$Z_1[y(t)](z) = \mathcal{Z}$$

$$Z_1\left([t](-2)^{[t]}\right)(z) = -z \frac{d}{dz} Z_1\left((-2)^{[t]}\right)(z) =$$

$$= -z \frac{d}{dz} \sum_{n=0}^{+\infty} \left(\frac{-2}{z}\right)^n = -z \frac{d}{dz} \frac{z}{z+2} =$$

$$= -z \frac{z+2-z}{(z+2)^2} = \frac{-2z}{(z+2)^2} \qquad |z| > 2$$

Quindi dall'equazione otteniamo

$$z\mathcal{Z} + 2\mathcal{Z} = \frac{-2z}{(z+2)^2}$$

$$\mathcal{Z} = \frac{-2z}{(z+2)^3}$$

Antitrasformando

$$\cancel{x(t)}\,a^{n} = \frac{1}{2\pi i}\int_{+\gamma_\rho(0)} \frac{-2z^{n}}{(z+2)^3}\,dz =$$

$$= \operatorname{Res}\left(\frac{2z^{n}}{(z+2)^3},\infty\right) = -\operatorname{Res}_{z=-2}\frac{+2z^{n}}{(z+2)^3} =$$

~~Per~~ la relazione tra zeri e poli $z=-2$ polo di ordine 3

$\Rightarrow$

$$a_n = -\frac{1}{2}\lim_{z\to -2} D^2(2z^n) = -\lim_{z\to -2} n(n-1)z^{n-2} =$$

$$= -n(n-1)(-2)^{n-2} = \frac{-n(n-1)(-2)^n}{4}$$

4) I caso a) $M/M/1$

$\lambda = 6/\text{giorno} \neq \frac{6}{8}/h = \frac{3}{4}/h$

$\mu = \frac{1}{E(S)} = 1\ /h$

$\rho = \frac{\lambda}{\mu} = \frac{3}{4} < 1$

b) $L = \frac{\rho}{1-\rho} = \frac{3/4}{1/4} = 3$ macchine

c) $W_\rho = \frac{\lambda}{(\mu-\lambda)\mu} = \frac{3/4}{(1-3/4)} = 3$ ore

II caso a) $M/M/1/4$

b) $L = \frac{\rho}{1-\rho} - \frac{\rho^{k+1}(k+1)}{1-\rho^{k+1}} = \frac{3/4}{1/4} - \frac{\left(\frac{3}{4}\right)^5 5}{1-\left(\frac{3}{4}\right)^5} \simeq 1{,}44$ macchine

c) ~~Po~~ $\pi_0 = \frac{1-\rho}{1-\rho^{k+1}} \simeq 0{,}328$

$$\pi_4 = \rho^k \cdot \pi_0 = \left(\frac{3}{4}\right)^4 0{,}378 = 0{,}104$$

$$W = \frac{L}{\bar{\lambda}}$$

$$\bar{\lambda} = \lambda\,(1-\pi_4) = 2{,}672\ /h$$

$W = 2{,}14$ ore

$$W_p = W - \frac{1}{\mu} = 1{,}14 \text{ ore}$$