RECENT TRENDS ON NONLINEAR PHENOMENA

Organized by

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at the

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Aims and scope

Nonlinear phenomena arise in many different contexts such as geometry, physics, mechanics, engineering, biology, life sciences, just to name a few. This conference aims to bring together leading academic scientists, researchers and research scholars to exchange and share their experiences and research results about all aspects of Nonlinear Phenomena.

Talks:

- Daniele Andreucci, *Università 'La Sapienza' di Roma*
- Vieri Benci, *Università degli Studi di Pisa*
- Fabrice Bethuel, *Université Pierre & Marie Curie - Paris 6*
- Michel Chipot, *Universitat Zurich*
- Donato Fortunato, *Università degli Studi di Bari*
- Jesús Hernández, *Universidad Autonoma de Madrid*
- Jean Mawhin, *Catholic University of Louvain*
- Patrizia Pucci, *Università degli Studi di Perugia*
- Vicentiu Radulescu, *King Abdulaziz University*
- Dusan Repovs, *University of Ljubljana*
- Biagio Ricceri, *Università degli Studi di Catania*
- Sandro Salsa, *Politecnico di Milano*
- Carlo Sbordone, *Università degli Studi di Napoli 'Federico II'*

Short Talks:

- Giovanni Anello, *Università degli Studi di Messina*
- Luigi D'Onofrio, *Università di Napoli 'Parthenope'*
- Antonio Greco, *Università degli Studi di Cagliari*
- Piotr Rybka, *University of Warsaw*
- Francesco Tulone, *Università degli Studi di Palermo*
- Binlin Zhang, *Heilongjiang Institute of Technology*

Talks

Optimal bounds and asymptotic behavior of porous-media like equations in non-compact domains

Daniele Andreucci
Università 'La Sapienza' di Roma, Italy

We investigate the asymptotic behavior for large times of solutions to porous-media like equations with variable coefficients set in an unbounded open region, with zero Neumann data prescribed on the boundary, which is unbounded too.

We identify the asymptotic profile; as a marked difference with the case of the Cauchy problem, such a profile is anisotropic due to the shape of the domain itself, and actually one-dimensional in space. We also estimate the rate of convergence to this profile, still in dependence of the geometry of the domain.

Generalized solutions in PDE's and Non-Archimedean Mathematics

Vieri Benci
Università degli Studi di Pisa, Italy

In many problems of Geometry and Mathematical Physics, the notion of function is not sufficient and it is necessary to extend it. Among people working in partial differential equations, the theory of distribution of Schwartz and the notion of **weak solution** are the main tools to be used when equations do not have classical solutions. However, there are problems which do not have solutions, not even in the space of distributions. As model problem you may think of the Yamabe equation

$$-\triangle u = u^{p-1},\ u > 0,\ p \geq \frac{2N}{N-2} \tag{YE}$$

with Dirichlet boundary conditions in a bounded star-shaped open set.

Moreover, there are equations which have more than one weak solution even when uniqueness is expected. Usually, these equations do not have classical solutions since they develop singularities and the conservation laws are violated. As an example let us consider the Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{BE}$$

A classical solution of the Cauchy problem, $u(t,x)$, is unique and, if it has compact support, it preserves the momentum $P = \int u\ dx$ and the energy $E = \frac{1}{2}\int u^2 dx$ as well as other quantities. However, at some time a singularity appears and the solution cannot be longer described by a smooth function. The notion of weak solution is necessary, but the problem of uniqueness becomes a central issue and the energy is not preserved.

Having these problems in mind, we construct a new class of functions called **ultrafunctions** in which the above problems have (generalized) solutions which behave as expected. In this construction, we apply the general ideas of Non Archimedean Mathematics and some techniques of Non Standard Analysis.

Branched transportation and singularities of Sobolev maps between manifolds

Fabrice Bethuel
Université Pierre & Marie Curie - Paris 6, France

Branched transportation plays an important role in the study of Sobolev maps between manifolds. After a brief introduction on the main issues in branched transportation theory, in particular irrigation, I will show how this field is connected to the question of sequentially weak density in Sobolev spaces between manifolds and eventually leads to the construction of obstructions to the later property.

Variational problems set on domains tending to infinity

Michel Chipot
Universitat Zurich, Switzerland

Let $\Omega_\ell = \ell\omega_1 \times \omega$, where $\omega_1 \subset \mathbb{R}^p$ and $\omega \subset \mathbb{R}^{n-p}$ are assumed to be open and bounded. We consider the following minimization problem:

$$E_{\Omega_\ell}(u_\ell) := \min_{u \in W_0^{1,q}(\Omega_\ell)} E_{\Omega_\ell}(u)\,,$$

where $E_{\Omega_\ell}(u) = \int_{\Omega_\ell} F(\nabla u) - fu$, F is a convex (or strongly convex) function and $f \in L^{q'}(\omega)$. We are interested in studying the asymptotic behavior of the solution u_ℓ as ℓ tends to infinity (joint work with A. Mosjic and P. Roy).

An abstract result for solitons and application to some nonlinear field equations

Donato Fortunato
Università degli Studi di Bari, Italy

We deal with an abstract existence theorem for solitons and we apply this theorem to some nonlinear field equations (nonlinear Klein-Gordon equation, nonlinear Schroedinger equation, nonlinear beam equation,..).

The results we state have been obtained in collaboration with V. Benci.

Positive and compact support solutions for some singular semilinear elliptic problems

Jesús Hernández
Universidad Autonoma de Madrid, Spain

We present some recent work on the existence of positive and compact support (dead core) solutions for some singular semilinear elliptic equations with zero Dirichlet boundary conditions. The nonlinearities arising in the equations are singular or, at least, non-Lipschitz, close to the origin. In the one-dimensional case it is possible to give a complete description of the solution set by using energy methods for ODEs, obtaining positive and continua of infinitely many compact support solutions. In the case of dimension $N > 1$, existence of non-negative solutions is proved by using Nehari manifolds or asymptotic bifurcation. Existence of compact support solutions is obtained with a Pohozaev identity, asymptotic estimates for the solutions and local comparison arguments. In this case the description is not complete. The second example is motivated by some questions concerning solutions to the linear Schrödinger equation with the infinite-well potential.

This is joint work with J.I. Díaz and Y. Ilyasov.

The multiplicity of solutions of relativistic-type systems with periodic nonlinearities

Jean Mawhin
Catholic University of Louvain, Belgium

Brezis and the author have shown in 2010, using variational methods, that the forced relativistic pendulum equation

$$\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' + a \sin u = e(t),$$

has at least one T-periodic solution when the forcing term e has a mean value zero on $[0,T]$. The existence of a second solution has been proved in 2012 by C. Bereanu and P. Torres, using a combination of the mountain pass lemma and lower-upper solutions techniques.

For the more general system

$$\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' + \nabla F_u(t,u) = e(t), \tag{1}$$

where u and e take values in $\mathbb{R}^n$, F is T_i-periodic in each u_i for some $T_i > 0$ $(i = 1, \ldots, n)$, and e has mean value zero on $[0,T]$, the existence of at least $n+1$ T-periodic solutions has been first proved by the author in 2012 using a reduction to some Hamiltonian form and Szulkin's multiplicity result for strongly indefinite functionals based upon relative Lusternik-Schnirel'man category. It contains Bereanu-Torres' result as a special case. The same conclusion has been obtained in 2013 by C. Bereanu and P. Jebelean from an abstract multiplicity result for convex, lower semicontinuous perturbations of a C^1 functional, obtained by combining a deformation lemma together with Ekeland's variational principle and classical Lusternik-Schnirelman category.

In this talk, which describes some joint work with P. Jebelean and C. Serban, we show that a suitable modification of the action functional and the use of a priori estimates allows reducing the problem to a C^1-functional invariant under some group action, to which a variant of Lusternik-Schnirelman's classical theorem which can be traced to Paul Rabinowitz can be applied.

The same simple approach also provides new multiplicity results when the periodicity of F with respect to the u_i is replaced by some asymptotic oscillatory behavior.

Combined effects in Kirchhoff fractional elliptic problems with lack of compactness

Patrizia Pucci
Università degli Studi di Perugia, Italy

The seminar will be focused on recent results concerning existence, multiplicity and asymptotic behavior of positive solutions of some Kirchhoff type problems, involving fractional integro-differential elliptic operators and presenting also difficulties due to intrinsic lacks of compactness, which arise from different reasons. The problems presented are highly nonlocal because of the presence of the fractional integro-differential elliptic operators and of the Kirchhoff coefficients. The proof techniques should therefore overcome the nonlocal nature of the problems as well as the lack of compactness, and the suitable strategies adopted depend of course on the problem under consideration.

Variational analysis on fractals

Vicentiu Radulescu
King Abdulaziz University, Jeddah, Saudi Arabia

In this talk, we report on some recent results in collaboration with Giovanni Molica Bisci. We are concerned with the qualitative analysis of solutions to some nonlinear elliptic problems on fractal domains. Our analysis includes the case of nonlinear terms with oscillatory behavior, either at the origin or at infinity. The approach combines variational arguments with the geometric properties of the Sierpinski gasket.

Elliptic problems on fractals

Dusan Repovs
University of Ljubljana, Slovenia

We discuss some existence results for a parametric Dirichlet problem defined on the Sierpinski fractal. More precisely, some critical point results for differentiable functionals are exploited, in order to prove the existence of positive eigenvalues for which the problem admits solution. The results are obtained in collaboration with Giovanni Molica Bisci.

A survey of some applications of certain minimax theorems

Biagio Ricceri
Università degli Studi di Catania, Italy

In this lecture, we intend to highlight a long series of consequences and applications of certain minimax theorems. Here are two samples of a very different nature.

Theorem 1. *Any non-empty, uniquely remotal and compact subset of a normed space is a singleton.*

Theorem 2. *Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain, with $n \geq 4$, let $a, b, \mu \in \mathbf{R}$, with $a \geq 0$ and $b > 0$, and let $p \in \left]0, \frac{n+2}{n-2}\right[$.*

Then, for each $\lambda > 0$ large enough and for each convex set $C \subseteq L^2(\Omega)$ whose closure in $L^2(\Omega)$ contains $H_0^1(\Omega)$, there exists $v^* \in C$ such that the problem

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = \mu|u|^{p-1}u + \lambda(u - v^*(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the corresponding energy functional.

Free-boundary problems of Stefan type: regularity issues and open problems

Sandro Salsa
Politecnico di Milano, Italy

The regularity (both of the solution and the free boundary) in Stefan type problem governed by general evolution equation is still far to be considered complete. Here we present some recent developments and indicate the main relevant open questions. Joint work with F. Ferrari.

BMO-solvability of the Dirichlet problem for elliptic equations. Sharp results

Carlo Sbordone
Università degli Studi di Napoli 'Federico II', Italy

We study the solvability of the Dirichlet problem

$$\begin{cases} \operatorname{div}(A(x)\nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega\,, \end{cases}$$

where $g \in L^p(\partial\Omega)$ or $g \in BMO(\partial\Omega)$.

Let Ω be the unit disc in $\mathbb{R}^2$ or the upper half plane $\mathbb{R}^2_+$ and let the matrix $A(x)$ be symmetric and uniformly elliptic:

$$\frac{|\xi|^2}{K} \leq \langle A(x)\xi, \xi\rangle \leq K|\xi|^2$$

for a $K > 1$, for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^2$.

As an example of our results we illustrate sharp self-improving solvability results, in terms of a quantiable absolute continuity of the elliptic measure of the operator $L = \operatorname{div}(A(x)\nabla\ )$.

Short talks

Elliptic problems with non Lipschitz nonlinearities: some recent results and open questions

Giovanni Anello
Università degli Studi di Messina, Italy

We will consider an autonomous sublinear elliptic problem with sign-changing nonlinearity and non lipschitz nonlinearity and we will present some existence and multiplicity results of nonnegative and positive solutions, and an existence result of least energy sign-changing solutions. Finally, some open questions will be discussed.

Bi-Sobolev homeomorphism with zero Jacobian almost everywhere

Luigi D'Onofrio
Università di Napoli 'Parthenope', Italy

In geometric function theory one of the most important properties is the fact that f maps sets of measure zero to sets of measure zero and that preimages of sets of measure zero have zero measure.

It was known already to Ponomarev [4] that it is possible to construct a Sobolev homeomorphism which maps a null set to a set of positive measure. Moreover, in [1], the Authors show that there are Lipschitz mappings which map a set of positive measure to a null set and thus $J_f = 0$ on this set of positive measure.

Recently, in [3] it was shown that it is possible to construct even a homeomorphism in the Sobolev space $W^{1,p}((0,1)^n, (0,1)^n)$ for any $p \in [1, n)$ such that $J_f = 0$ a.e. It follows that such a mapping simultaneously sends a null set to a set of full measure and a set of full measure to a null set.

In this work in collaboration with Stanislav Hencl and Roberta Schiattarella (see [2]), we address the issue of the possible Sobolev regularity of the inverse of this pathological homeomorphism. In particular we show that for $n \geq 3$, it is possible to construct a homeomorphism f in the Sobolev space $W^{1,1}((0,1)^n, (0,1)^n)$ such that $f^{-1} \in W^{1,1}((0,1)^n, (0,1)^n)$, $J_f = 0$ a.e. and $J_{f^{-1}} = 0$ a.e. Combining some non trivial known results, we prove that such a pathological homeomorphism cannot exist in dimension $n = 2$ or in higher dimension with $W^{1,n-1}$ regularity of f. Moreover, we discuss on the Sobolev regularity of such pathological homeomorphism and the Sobolev regularity of the inverse map.

References

[1] Astala K., Iwaniec T. and Martin G., *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.

[2] D'Onofrio L., Hencl S. and Schiattarella R., *Bi-Sobolev homeomorphism with zero Jacobian almost everywhere*, Calc. Var., 51 (2014), 139–170.

[3] Hencl S., *Sobolev homeomorphism with zero Jacobian almost everywhere*, J. Math. Pures Appl., 95 (2011), 444–458.

[4] Ponomarev S., *Examples of homeomorphisms in the class $ACTL^p$ which do not satisfy the absolute continuity condition of Banach* (Russian), Dokl. Akad. Nauk USSR, 201 (1971), 1053–1054.

Fractional convexity maximum principle

Antonio Greco
Università degli Studi di Cagliari, Italy

The celebrated *convexity maximum principle* was proved by Nick Korevaar in the 80's to answer a question posed by his advisor, prof. Robert Finn, concerning convexity of capillary surfaces in convex pipes. Korevaar's idea gave birth to a number of subsequent contributions, especially due to Kawohl and Kennington. Instead of arguing by contradiction, Porru and the speaker in 1993 gave an alternative proof based on the construction of a suitable elliptic inequality, in the spirit of Larry Payne's P-functions. This talk deals with a possible extension of the convexity maximum principle to continuous solutions of equations involving the fractional Laplacian. Applications are discussed.

Viscosity solutions for closed curves driven by a singular weighted mean curvature flow

Piotr Rybka
University of Warsaw, Poland

Earlier, we constructed variational solutions for closed curves driven by a singular wmc flow. Actually, we considered curves which are small perturbations of a scaled Wulff shape, i.e. a ball in the norm specified by the anisotropy function.

Recently, M.-H.Giga and Y.Giga have developed the viscosity theory for singular parabolic problems in one-dimension. In order to make this theory work, we treat our evolving curve as graph over a suitable reference manifold and we rewrite the wmc flow as a singular parabolic pde for an evolving graph.

Using the methods of the viscosity theory we study issues, which were not tractable with the tools we used earlier. In particular, we show uniqueness of solutions and address the corner persistence problem.

Multiple Haar and Walsh series and generalized integrals

Francesco Tulone
Università degli Studi di Palermo, Italy

We consider various aspects of the problem of recovering the coefficients of multidimensional Haar and Walsh series from their sums. The theory of non-absolute generalizations of the Lebesgue integral, needed to solve the problem of recovering, by generalized Fourier formulas, the coefficients of orthogonal series from its sum, was initiated by Denjoy who in the classical trigonometric case introduced an integration process so powerful that the sum of any everywhere convergent trigonometric series is integrable in the sense of this integral and the coefficients of the series are Denjoy-Fourier coefficients. Later an easier approach based on Perron and Kurzweil-Henstock methods was developed by several authors. The same problem was considered also for series with respect to some other orthogonal systems including system of Haar wavelets, Walsh and Vilenkin systems and more general systems of characters of zero-dimensional compact abelian groups. In the one-dimensional case the problem was solved using various types of generalized integrals, including dyadic Denjoy, dyadic Perron and dyadic Kurzweil-Henstock integrals. In the multidimensional case a solution for the same series depends on the type of convergence (regular, cubic, rectangular etc.).

Here we concentrate on n-dimensional Haar and Walsh series which are rectangular convergent outside exceptional sets from some class of U-sets, without assuming a priori integrability of the sum in any prescribed sense, and solve the coefficients problem by finding an appropriate integral to be used in generalized Fourier formulas. As it was in one-dimensional case, the method is based on reducing the coefficients problem to the one of recovering a function from its derivative with respect to the appropriately chosen dyadic derivation basis. The difficulties which should be overcome in applying this method are related to the fact that the primitive we want to recover is differentiable not everywhere but outside some exceptional set which is not countable in a dimension greater then one. This is the reason why the dyadic Kurzweil-Henstock integral, which solves the problem in the one-dimensional case, turns out not to be strong enough to recover the primitive in multidimensional case under our assumptions. Because of this we have to introduce a suitable Perron-type integral defined by major and minor functions having a special continuity property, namely the so-called local Saks continuity. We show that each n-dimensional Walsh series which converges everywhere outside a U-set of the type we consider here, is the Fourier series of its sum in the sense of this Perron-type integral. The same result, with some additional assumption on the behavior of the coefficients, is obtained for Haar series.

Multiplicity results for some classes of fractional p–Laplacian problems

Binlin Zhang
Heilongjiang Institute of Technology, China

In this talk we are concerned with some recent multiplicity results for the following problems involving the fractional p–Laplacian: Kirchhoff type, Schrödinger type and Schrödinger–Kirchhoff type. By using variational methods, we investigate multiple solutions, especially infinitely many solutions, for the aforementioned type equations. As a particular case, we present some multiplicity results for the fractional p–Laplacian equations.